M-math 2nd year Final Exam
Subject : Probability Theory

Time : 2.30 hours Max.Marks 50.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. All random variables or stochastic processes mentioned below will be on this probability space.

(1) (a) Let $\{B_t\}$ be a standard Brownian motion. Prove that, with probability 1,
$$\lim_{n\to\infty} \frac{1}{n} B_{2n} = 0.$$
(7)

(b) Let $\{X_t\}_{t\in[0,\infty)}$ be a stochastic process. Show that the following are equivalent.
 (i) $\{X_t\}$ is a Brownian motion.
 (ii) $\{X_t\}$ is a continuous centred Gaussian process with $Cov[X_s, X_t] = s \wedge t, \forall s, t \geq 0$. (4+4)

(c) Let $\{B_t\}$ be a Brownian motion. Using part (b) or otherwise, show that $\{\frac{1}{2}B_{4t}\}$ is also a Brownian motion. (5)

(2) Let (B_t) be a standard one dimensional Brownian motion.
(a) Let X be an $N(0,1)$-distributed random variable, which is independent of $\{B_t\}$. For any $t \in [0,1]$ show that
$$P(\sqrt{1-t}\,|X| \leq |B_t|) = \frac{2}{\pi} \arcsin\left(\sqrt{t}\right).$$
(10)

(b) Let $\tau_b := \inf\{s > 0 : B_s = b\}$. Show that for $b > 0$,
$$E(e^{\lambda\tau_b}) = e^{-b\sqrt{2\lambda}}.$$
(7)

(c) Show that for any $a > 0, t > 0$
$$P\{\sup\{B_s, 0 \leq s \leq t\} > a\} = 2P\{B_t > a\}.$$
(8)

(3) Let (B_t) be any one dimensional Brownian motion, $(\mathcal{F}_t)$ its natural filtration and τ a finite stopping time.
(a) Show that $(B_{t+\tau} - B_\tau)_{t\geq 0}$ is a standard Brownian motion independent of $\mathcal{F}_\tau$.
(b) Using part a) or otherwise show that the strong Markov property holds at τ.

(10+ 5)